Example find $f'(h)$ if

$$f(h) = \underbrace{4h}\,\underbrace{\sin(4h - \cos(h))}$$

$$f'(h) = 4\sin(4h - \cos(h)) + 4h\left[\sin(4h - \cos(h))\right]'$$

$$= 4\sin(4h - \cos(h)) + 4h \cdot \cos(4h - \cos(h)) \cdot (4h - \cos(h))'$$

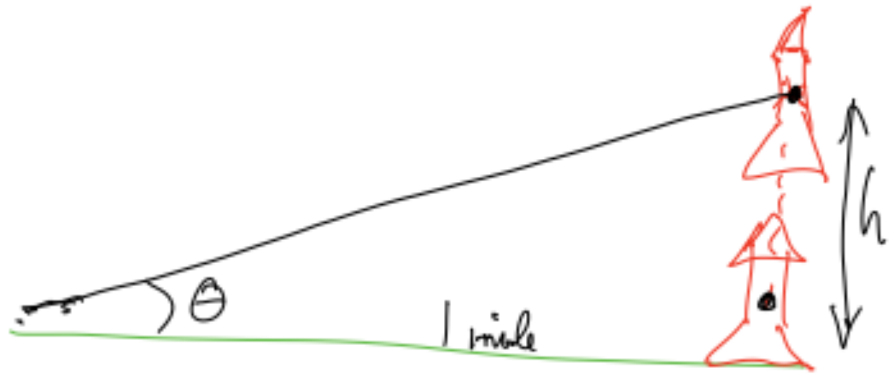
$$= \boxed{4\sin(4h - \cos(h)) + 4h\cos(4h - \cos(h))(4 + \sin(h))}$$

Related Rates

Example A rocket is being launched a mile away, and an observer measures that when the rocket is at an angle of 4° above the horizontal, the angle is changing at a rate $0.18^\circ/\text{sec}$.

Questions: How far off the ground is the rocket at this moment?

How fast is it going?



$$0.18^\circ/\text{sec} = \frac{d\theta}{dt}$$

$$\theta = 4^\circ$$

$h = ?$

$\frac{dh}{dt} = ?$ ← speed

$$\boxed{\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{h}{1\text{ mile}}}$$

$$\tan 4^\circ = \frac{h}{1\text{ mile}} \Rightarrow h = (\tan(4^\circ))\,1\text{ mile}$$

$$\Rightarrow h = 0.069927 \text{ miles}$$

$$\Rightarrow h = (0.069927\ \cancel{\text{mile}})\frac{5280\text{ ft}}{1\ \cancel{\text{mile}}}$$

$$= \boxed{369.2\text{ ft.}}$$

Take deriv

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{h}{5280\text{ ft.}}\right)$$

$$\frac{1}{\cos^2\theta}\cdot\frac{d\theta}{dt} = \frac{1}{5280\text{ ft.}}\cdot\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{5280\text{ft}}{\cos^2(\theta)} \frac{d\theta}{dt}$$

(θ must be in radians!)

$$\frac{d\theta}{dt} = \frac{.18^\circ}{\text{sec}} \cdot \frac{\pi \text{ radians}}{180^\circ}$$

given

$$= \frac{.001\pi \text{ radians}}{\text{sec}}$$

$$= \frac{.00314 \text{ radians}}{\text{sec.}}$$

$$\Rightarrow \frac{dh}{dt} = \frac{(5280\text{ft})}{\cos^2(4^\circ)} \cdot \boxed{\frac{.00314 \text{ rad.}}{\text{seconds}}}$$

$$\cos^2\left(\frac{4\pi}{180}\right)$$

$$= \boxed{16.7 \text{ ft/sec}}$$

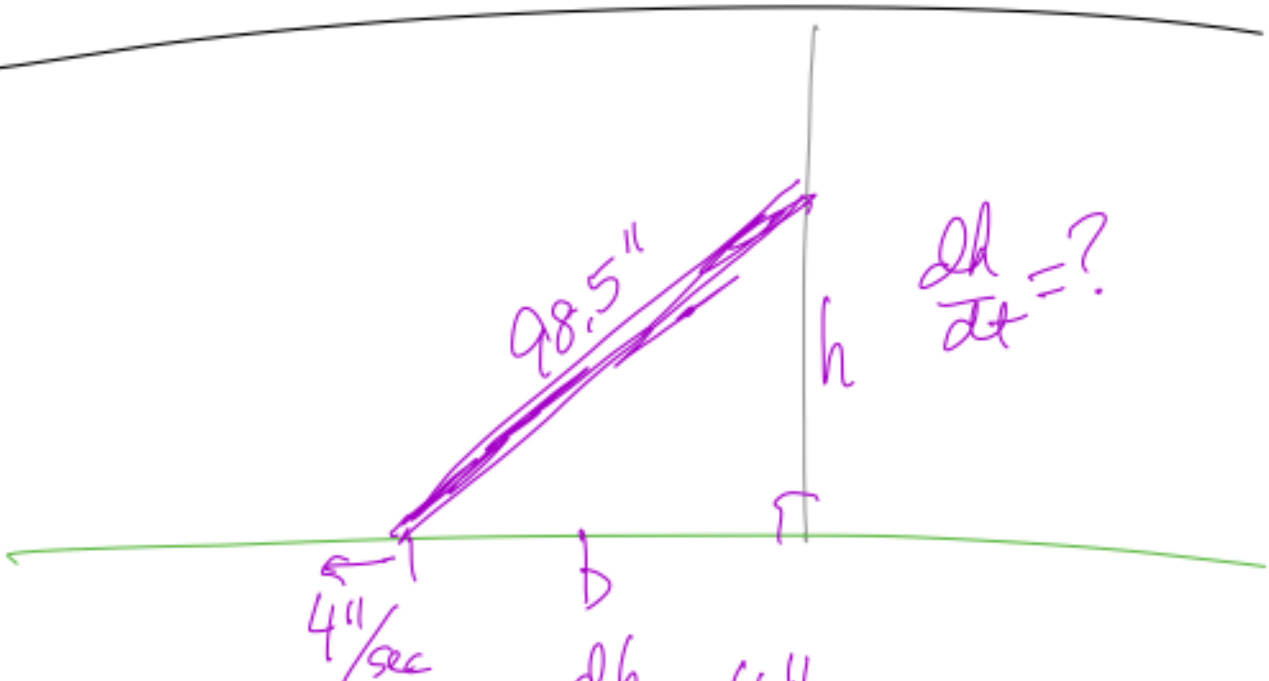
Example. A ladder is leaning up against a wall.

The bottom of the ladder is moving away from the wall at a rate of 4.00 inches/sec. The Length of the ladder is 98.5 inches.

① Find a formula for the speed at which the ladder is moving down when its height on the wall is h inches.

② How fast is the top of the ladder moving when $h = 5\text{ft} = 60.0$ inches?

③ How fast is the top of the ladder moving when $h = 1\text{ ft} = 12.0$ inches?



$\frac{db}{dt} = \frac{4''}{\text{sec}} = \text{constant}.$

Equations: $b^2 + h^2 = (98.5)^2$

$$\frac{d}{dt}\left(b^2+h^2\right) = \frac{d}{dt}\left((98.5)^2\right) = 0$$

$$2b \cdot \frac{db}{dt} + 2h\frac{dh}{dt} = 0$$

$$\frac{db}{dt} = 4''/\text{sec} \qquad \frac{dh}{dt} = ?$$

h = variable

$$b = \sqrt{(98.5)^2 - h^2}$$

$$2\sqrt{(98.5)^2 - h^2}\left(4''/\text{sec}\right) + 2h\frac{dh}{dt} = 0$$

① $$\boxed{\frac{dh}{dt} = \frac{-8\sqrt{(98.5)^2 - h^2}\ \text{inch}/\text{sec}}{2h}}$$

② $h = 60.0$ in

$$\frac{dh}{dt} = \frac{-8\sqrt{(98.5)^2 - 60^2}}{120}\ \text{in}/\text{sec} = -5.207\ \text{in}/\text{sec}$$

③ $h = 12.0$ in.

$$\frac{dh}{dt} = \frac{-8\sqrt{(98.5)^2 - 12^2}}{24}\ \text{in}/\text{sec} = -32.588\ \text{in}/\text{sec}$$

④ $h = \cancel{\text{two}}\ 0.01\text{ in}$ [what is $\frac{dh}{dt}$ when $h = 0.01\text{ in}$]

$$\frac{dh}{dt} = \frac{-8\sqrt{98.5^2 - 0.01^2}}{0.02}\ \text{in/sec}$$

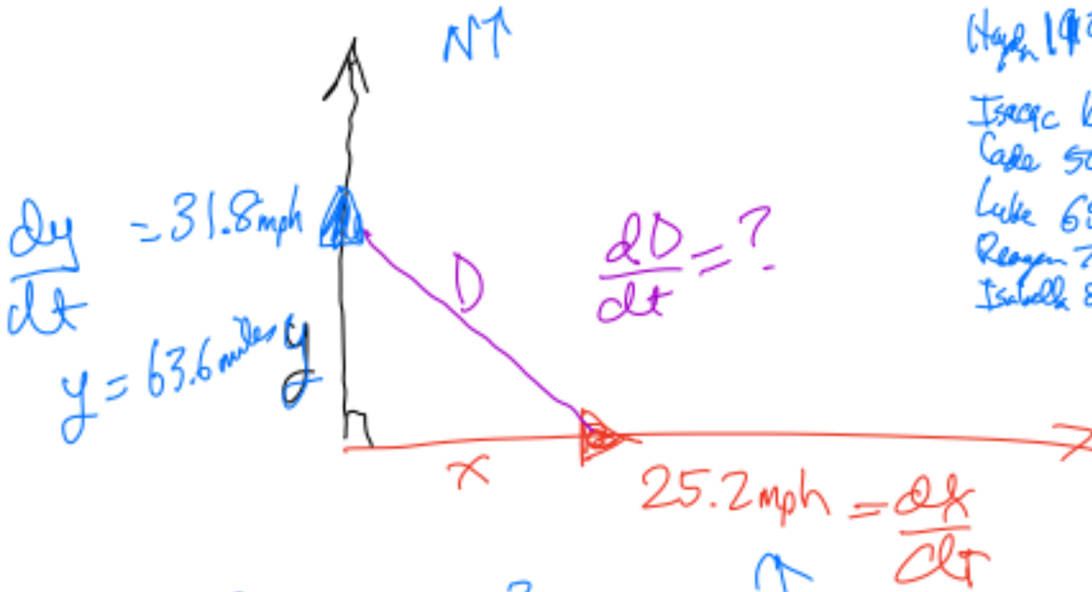
$$\frac{dh}{dt} = -31{,}399.99\ \text{in/sec}$$

$$= -3283.33\ \text{ft/sec}$$

$$= \boxed{2{,}238\ \text{mph}}$$

Example A ship is traveling north at a speed of 31.8 mph, and another ship is traveling east at a speed of 25.2 mph. If they start at the same point & time, how fast will the distance between them be changing after two hours?

Hayden 100 mph
Isaac 65 mph
Cale 50 mph
Luke 65
Reagan 75
Isabella 80



after 2 hrs,
$x = 50.4$ miles

$$x^2 + y^2 = D^2$$

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}\left(D^2\right)$$

$$2\underbrace{x}_{50.4}\underbrace{\frac{dx}{dt}}_{25.2} + 2\underbrace{y}_{63.6}\underbrace{\frac{dy}{dt}}_{31.8} = 2D\underbrace{\frac{dD}{dt}}_{?}$$

$$D = \sqrt{x^2 + y^2}$$

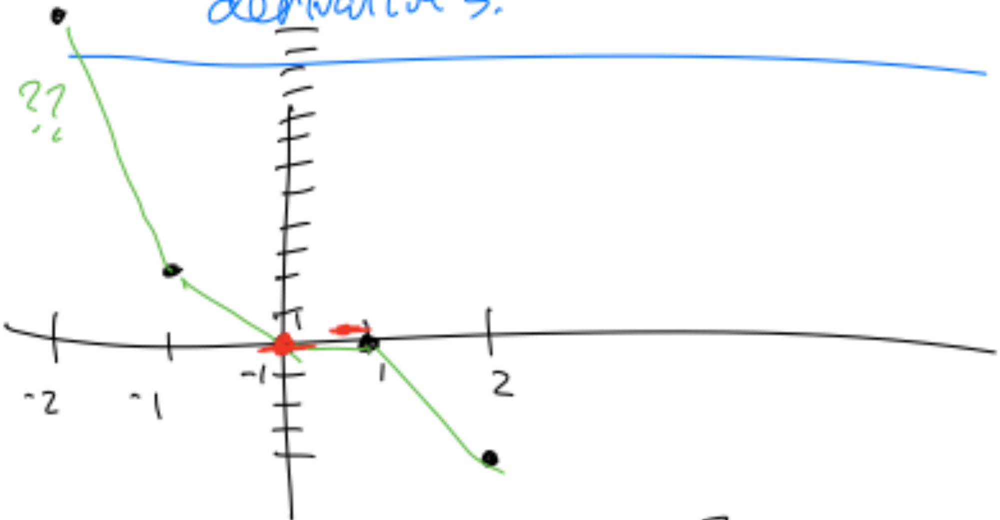
$$D = \sqrt{50.4^2 + 63.6^2}$$

$$= 81.1 \text{ miles}$$

$$2(50.4)(25.2) + 2(63.6)(31.8) = 2(81.1)\frac{dD}{dt}$$

$$\Rightarrow \frac{dD}{dt} = \frac{2(50.4)(25.2) + 2(63.6)(31.8)}{2(81.1)} = \boxed{40.6 \text{ mph}}$$

Next application: Geometry of Graphs using derivatives.



Example $y = x^2 - x^3$.

x	y
-2	$4-(-8)=12$
-1	$1-(-1)=2$
0	0
1	$1-1=0$
2	$4-8=-4$

We can get more information from the derivative:

$$\frac{dy}{dx} = 2x - 3x^2$$ ← When is this positive, negative?

factor. $= x(2-3x) = 0$

When $x = 0$ or $\frac{2}{3}$. where slope $=0$.

$x=0 \quad y=0$

$x=\frac{2}{3} \quad y=\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3=\frac{4}{9}-\frac{8}{27}$

$\frac{12-8}{27}=\frac{4}{27}\approx .15$

Analyze the sign of $\frac{dy}{dx} = x(2-3x)$

$\frac{dy}{dx} =$ - - - - - - - 0 + + + + 0 - - - - - -